Two-dimensional subsonic and sonic flow past thin bodies

By J. B. HELLIWELL
The Royal College of Science and Technology, Glasgow

and A. G. MACKIE
St. Salvator's College, University of St. Andrews

(*Received* 28 *February* 1957)

SUMMARY

Hodograph methods are applied to determine the flow at high subsonic and sonic velocities past two-dimensional, thin, symmetrical bodies. The boundary value problem for the determination of the stream function Ψ, which in the present theory is a solution of Tricomi's equation, is simplified by the assumption of a free stream breakaway at sonic velocity from the shoulder of the body. A solution is obtained in terms of Bessel functions.

In §§ 2 and 3 the flow past a wedge of small angle is discussed and expressions are obtained for the pressure on the nose, the drag coefficient and the width of the wake. A comparison with the corresponding results in the case of sonic velocity derived by the more complex analysis of Guderley & Yoshihara (1950) shows that the present simpler theory yields very similar values for the pressure over the nose.

In § 4 the flow at sonic velocity past a profile which is a first-order perturbation upon a wedge profile is analysed on the basis of the same free streamline theory. The flow pattern is obtained past an *arbitrarily specified* body by an application of the Hankel inversion theorem and an expression is deduced for the drag.

1. Introduction

In the determination of the steady two-dimensional flow of a non-viscous gas past a symmetric obstacle various methods, both approximate and exact, are available provided the velocity of the gas throughout the flow field is bounded away from the local velocity of sound, either above or below. But in the neighbourhood of sonic velocity the approximate methods in general break down because of the change in type from elliptic to hyperbolic of the relevant differential equation at this point. A powerful method of overcoming some of the difficulties is the hodograph transformation. By means of this, the stream function Ψ can be written as a function of the velocity variables and the resulting differential equation is linear. Further it permits a solution by separation of variables in terms of trigonometric or hyperbolic functions and of certain hypergeometric functions usually denoted by $\psi_n(\tau)$. Here n is a parameter, real or complex, and $\tau = q^2/q_m^2$ where q is the velocity of the gas and q_m is the maximum velocity attainable by the gas when subject to adiabatic expansion. A description of the main properties of $\psi_n(\tau)$ has been given by Lighthill (1947). Suppose now that the velocity of the gas is nearly sonic everywhere and that the polar velocity angle θ is small. Then the cartesian components of velocity U and V (where $V = 0$ at infinity upstream of the obstacle) can be written as

$$U = a^* - u',$$
$$V = v',$$

where u' and v' are small. Here a^* is the velocity of the gas when the Mach number M is 1. The equation for Ψ can now be approximated by

$$\frac{\partial^2 \Psi}{\partial u'^2} + \frac{\gamma+1}{a^*} u' \frac{\partial^2 \Psi}{\partial v'^2} = 0,$$

where γ is the adiabatic index of the gas.

By defining dimensionless variables $u = (\gamma+1)u'/a^*$, $v = (\gamma+1)v'/a^*$, this becomes

$$\frac{\partial^2 \Psi}{\partial u^2} + u \frac{\partial^2 \Psi}{\partial v^2} = 0,$$

the well-known equation of Tricomi.

Roughly speaking there are two main methods of applying the hodograph transformation to the problem of flow past a body. One is to use the solutions $\psi_n(\tau)$ of the exact equations and, with analogous problems in incompressible flow as a guide, to write the stream function as a series of the form $\Psi = \sum a_n \psi_n(\tau) \sin n\theta$. In general the body shape which this solution gives has to be determined subsequently; the coefficients a_n cannot be specified for a body of arbitrary shape. This is a very unsatisfactory feature of the method but the difficulties may be overcome in the cases where the streamline $\Psi = 0$, the dividing streamline part of which follows the body contour, consists of portions on which either q or θ is constant. In general such patterns in incompressible flow are preserved in the generalization to allow for compressibility. Thus, for example, the problem of flow past a wedge when the streamline breaks away from the shoulder with constant velocity equal to the velocity at infinity upstream can be solved exactly for any wedge angle and any velocity at infinity, provided $M_1 \leqslant 1$. The suffix 1 will in future refer to conditions at infinity upstream.

An alternative method, much favoured by American workers in this field, is to use the Tricomi approximation and to set up the problem directly in the hodograph plane. An essential requirement in this approach is the determination of the singularity in the hodograph plane corresponding to the velocity at infinity. A singularity is clearly located here, since all the streamlines both originate from and return to this point. The singularity is consequently 'of doublet type' and this may also be surmised by consideration of the corresponding singularity in incompressible theory. However, while this singularity has been successfully identified by a number of authors, the same sort of difficulties arise with the other boundary conditions as previously. For a given boundary in the physical plane it is not in general possible to determine the relation holding between the velocity variables on this boundary and this means that the shape of the streamline $\Psi = 0$ in the *hodograph* plane cannot be predetermined. However, as before, when either the magnitude or direction of the velocity vector is constant on $\Psi = 0$, a solution of the problem is forthcoming.

Sections 2 and 3 of the present paper are concerned with discussion of flow past a wedge of small angle when the velocity at infinity is subsonic or sonic. For such a problem in incompressible flow when the velocity at infinity is q_1, the standard methods of Kirchhoff and Helmholtz lead to a solution in which the flow breaks away from the shoulder at $q = q_1$ and the free streamline which starts at the shoulder retains this constant velocity to infinity downstream. Throughout the flow field $q \leqslant q_1$. It is not desirable to obtain a similar solution for the case of compressible flow when the velocity at infinity upstream is subsonic because experimental evidence shows that, for reasonably high values of M_1, the gas is accelerated up the wedge side until it attains sonic velocity at the shoulder. We shall therefore select a flow pattern which exhibits this feature and the appropriate solution of Tricomi's equation will then be found. The following model will be adopted. The gas comprising the dividing streamline $\Psi = 0$ starts from infinity upstream and moves in a straight line up to the stagnation point at the tip of the wedge. It then accelerates up the wedge side, reaching sonic velocity at the shoulder. This velocity is retained until the streamline becomes parallel again to the free stream. It then remains straight, decelerating from sonic velocity to the velocity at infinity. We note that the streamline $\Psi = 0$ comes within the general category we have mentioned in that it consists of portions along which either the magnitude or the direction of the velocity vector remains constant.

This model has the advantage of yielding the relatively simple analytical solution developed in the next section while retaining the essential physical characteristics of flow past a wedge. In practice it might be expected that the Mach number of the streamline separating from the shoulder will be somewhat greater than 1, but the effect of this, particularly in the region upstream of the shoulder, will be small and may be neglected. A further advantage is that this is the type of model which Roshko (1954) adopted in studies of flow of incompressible fluid past bluff bodies. The fact that Roshko obtained such good agreement with experimental results using this 'notched hodograph' model suggests that this is a good basis on which to attempt a solution. Of course, in the limit as $M_1 \to 1$, we obtain the problem of the wedge in a sonic stream with a free streamline at sonic velocity extending from the shoulder to infinity. As has been mentioned, this particular problem can be solved exactly, that is, in terms of the exact differential equation for the stream function as distinct from the transonic approximation, but the leading term of this exact solution for small wedge angles is the limiting case of the problem considered here as $M_1 \to 1$. The expression for the stream function coincides with that of Imai (1952) who used a different approach. Imai based his solution on the incompressible flow and then replaced powers of the velocity by associated Bessel functions. His results, and those obtained by Mackie & Pack (1955) are discussed in § 3.

This limiting case is important because it gives a fairly simple representation of a wedge in a sonic stream and therefore it can be compared with the solution of Guderley & Yoshihara (1950). This solution was derived to represent the flow of a gas for which $M_1 = 1$ past a wedge profile of small angle up to the limiting Mach wave emanating from the shoulder. Downstream from the limiting characteristic any disturbance introduced will not affect the flow before it and consequently the continuation of the solution can be performed by the method of characteristics in a way which is determined by the subsequent profile of the object (parallel sides, diamond shape, etc.). When $M_1 = 1$ a new difficulty is encountered in the determination of the correct singularity in the hodograph plane. Whereas a subsonic singularity is isolated, occurring as it does in the elliptic region of the hodograph plane, the singularity is now at the origin and will be propagated along the characteristic in the hyperbolic part of the plane which starts from the origin. This characteristic maps into the limiting Mach wave in in the physical plane and care must be taken to choose a singularity which does not map this characteristic in the hodograph plane to infinity in the physical plane. The correct determination of the singularity and the subsequent addition of non-singular terms to satisfy the boundary conditions is a matter of considerable complexity and to keep the analysis manageable Guderley & Yoshihara found a certain amount of asymptotic approximation unavoidable. Thus it is of interest to compare the mathematically much simpler model of the 'free streamline' theory with the solution of Guderley & Yoshihara. The pressure (and hence velocity) distribution on the wedge side shows very little difference between the two solutions. A specific comparison is made in § 3.

The fact that the discrepancy is small is not surprising on physical grounds. For both solutions describe a flow with a stagnation point at the tip and sonic velocity at the shoulder of the wedge, while the flow at a great distance from the wedge is uniform with $M = 1$. The difference occurs in the form of solution immediately downstream of the shoulder and the upstream influence of this is of necessity small in a near sonic stream. Thus the principal difficulty in the work of Guderley & Yoshihara, which is the correct determination of the flow between the sonic line and the limiting characteristic, does not arise here since the whole physical plane (exclusive, of course, of the wake) is mapped into the elliptic part of the hodograph plane bounded by the sonic line. The absence of a supersonic region means further that no limit lines can appear. The occurrence of limit lines is an ever present danger in the hodograph theory of supersonic flow as their images are not readily detected in the hodograph plane but they render the transition to the physical plane meaningless.

The satisfactory representation thus obtained of the flow pattern upstream of the shoulder justifies an extension which is carried out in § 4. A profile is considered which is a first-order perturbation on a wedge profile. The corresponding flow past this profile can be obtained by an application of the Hankel inversion theorem. An unusual advantage of this solution is that it can be obtained for any given first-order perturbation in the *physical plane* and does not involve the *a posteriori* determination of the bounding streamline. An analytic expression for the drag on such a body is given in the general case.

2. The wedge in a subsonic stream

Before proceeding we shall summarize briefly the relationships holding between the physical variables. It should be remembered that in many cases these relations hold only within the limits of the transonic approximation.

We have already defined u and v. The pressure and density are p and ρ respectively. The Mach number M is given by

$$1-M^2 = u.$$

The polar angle θ is simply related to v by means of the equation $v = (\gamma+1)\theta$. The equation of continuity and the condition for irrotational flow may be written respectively as

$$uu_x - v_y = 0, \qquad u_y + v_x = 0.$$

Inverting these, we get the hodograph equations

$$uy_v - x_u = 0, \qquad x_v + y_u = 0. \tag{1}$$

These lead immediately to Tricomi's equation

$$y_{uu} + uy_{vv} = 0,$$

in which $u > 0$ represents subsonic flow. Thus y satisfies the same differential equation as Ψ and it may be shown that Ψ is a simple multiple of y. Similarly Φ, the velocity potential, is a multiple of x and in much of what follows we shall work with x and y as dependent variables instead of Φ and Ψ. In subsequent work with the flow nowhere supersonic we shall find it convenient to use a new variable r defined by

$$r = \tfrac{2}{3}u^{3/2}. \tag{2}$$

Tricomi's equation may then be solved according to the usual procedure of separation of variables and simple solutions are obtained of the type

$$y = r^{1/3}e^{\pm\lambda v}\mathscr{C}_{\pm 1/3}(\lambda r),$$

where $\mathscr{C}_{\pm 1/3}(\lambda r)$ is any linear combination of Bessel functions of order $\frac{1}{3}$ and λ may be real or imaginary.

The flow pattern described in the previous section is shown in figure 1. The figure is largely self-explanatory. Because of symmetry only the flow in the upper half-plane $y \geqslant 0$ need be considered. The dividing streamline $\Psi = 0$ is the line $EOBCD$. Along EO the velocity decreases from its (subsonic) value at infinity to zero at O, the tip of the wedge. Along the wedge side OB we have $v = v_0 = (\gamma + 1)\delta$ where δ is the semi-angle of the wedge. On BC the flow is sonic and along CD, $v = 0$, while the velocity decreases from sonic to its value at infinity upstream.

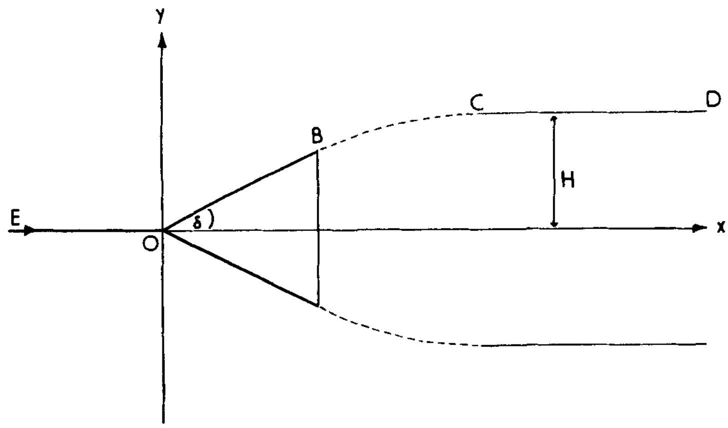

Figure 1. Physical plane.

The boundary value problem is now set up in the hodograph plane which is shown in figure 2. $OBCDEO$ is the line $y = 0$ and two other lines of constant y are also sketched. There is a singularity at the point where D and E coincide and in addition we must have

$$y = 0 \quad \text{on } v = 0, \quad u \geqslant 0,$$
$$y = 0 \quad \text{on } v = v_0, \quad u \geqslant 0,$$
$$y = 0 \quad \text{on } u = 0, \quad 0 < v < v_0.$$

The stagnation condition at the tip gives, in accordance with the linearization principle,

$$y = 0, \quad x = 0 \quad \text{as } u \to \infty.$$

Finally if the wedge is of unit length we must have

$$x = 1 \quad \text{at } v = v_0, \quad u = 0.$$

It should be noted that the problem now formulated is very similar to that considered by Cole (1951) in which the boundary condition along $u = 0$ is not $y = 0$ but $\partial y/\partial u = 0$. This results from the condition he imposes that the sonic line be straight and normal to the direction of the flow at infinity. A solution of the present problem may be obtained by a procedure similar to that carried out in §4 of Cole's paper. It turns out that the required solution is of the type Γ_{11} in Cole's notation which he rejects. This is

$$y = Ar^{1/3}\int_0^\infty \frac{\sinh\lambda(v_0-v)}{\sinh\lambda v_0} J_{1/3}(\lambda r)J_{1/3}(\lambda r_1)\lambda\, d\lambda. \tag{3}$$

Here A is a constant which will be fixed by the condition that the wedge is of unit length, r_1 is also a constant and is the value of r corresponding to $u = u_1$, the value of u at infinity upstream.

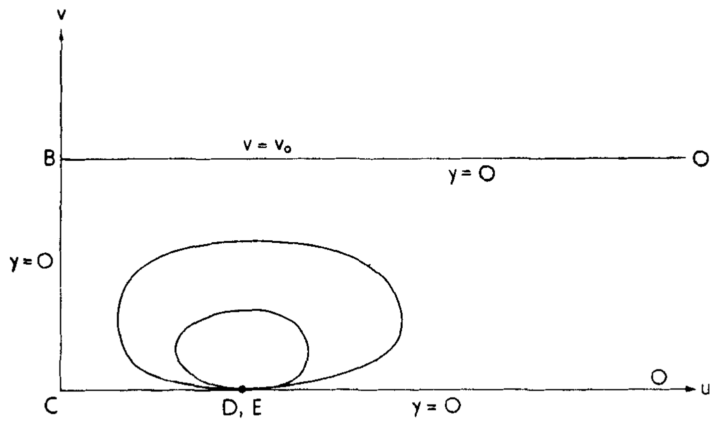

Figure 2. Hodograph plane.

Equations (1) now enable the corresponding x coordinate to be found. After a little analysis this is seen to be

$$x = A(\tfrac{3}{2})^{1/3}r^{2/3}\int_0^\infty \frac{\cosh\lambda(v_0-v)}{\sinh\lambda v_0} J_{-2/3}(\lambda r)J_{1/3}(\lambda r_1)\lambda\, d\lambda \tag{4}$$

The arbitrary constant which appears in this determination of x is easily shown to be zero because of the stagnation condition $x = 0$ as $r \to \infty$.

Since $x = 1$ at the shoulder where $v = v_0$ and $r = 0$ we obtain from (4)

$$1 = \frac{A(\frac{3}{2})^{1/3}2^{2/3}}{\Gamma(\frac{1}{3})}\int_0^\infty \frac{\lambda^{1/3}J_{1/3}(\lambda r_1)}{\sinh\lambda v_0}\, d\lambda.$$

On expanding the Bessel function, setting $\lambda v_0 = t$ and interchanging orders of summation and integration we get

$$1 = \frac{A6^{1/3}}{\Gamma(\frac{1}{3})}\sum_{n=0}^{\infty}\frac{(-1)^n r_1^{2n+1/3}}{v_0^{2n+5/3}2^{2n+1/3}}\frac{1}{n!\,\Gamma(n+\frac{4}{3})}\int_0^\infty \frac{t^{2n+2/3}}{\sinh t}\, dt.$$

Making use of a well-known representation of the Riemann zeta function $\zeta(s)$, we obtain finally after some algebra

$$1 = \frac{A3^{1/3}r_1^{1/3}}{\pi^{1/2}\Gamma(\frac{1}{3})}\sum_{n=0}^{\infty}\frac{(-1)^n r_1^{2n}\Gamma(n+\frac{5}{6})(2^{2n+5/3}-1)\zeta(2n+\frac{5}{3})}{2^{2n}n!\,v_0^{2n+5/3}} \tag{5}$$

This equation determines the constant A.

In the subsequent work a series representation for x on the side of the wedge will be useful. To obtain this we express $\operatorname{cosech}\lambda v_0$ in its partial fraction expansion

$$\operatorname{cosech}\lambda v_0 = \frac{1}{\lambda v_0} + 2\sum_{n=1}^{\infty} \frac{(-1)^n \lambda v_0}{(\lambda v_0)^2 + (n\pi)^2}.$$

Substituting this value in (4) with $v = v_0$ we finally derive, after some manipulation, the following series

$$x = \frac{A(\frac{3}{2})^{1/3} r_1^{-1/3}}{v_0} \left\{1 + \frac{2\pi r^{2/3} r_1^{1/3}}{v_0} \sum_{n=1}^{\infty} (-1)^n n I_{-2/3}\left(\frac{n\pi r}{v_0}\right) K_{1/3}\left(\frac{n\pi r_1}{v_0}\right)\right\}, \quad (6)$$

for $0 < r < r_1$, and

$$x = -\frac{2\pi A(\frac{3}{2})^{1/3} r^{2/3}}{v_0^2} \sum_{n=1}^{\infty} (-1)^n n I_{1/3}\left(\frac{n\pi r_1}{v_0}\right) K_{-2/3}\left(\frac{n\pi r}{v_0}\right), \quad (7)$$

for $0 < r_1 < r$.

A further series representation for x can be obtained for the portion of the streamline parallel to the uniform upstream flow. In this case $v = 0$ and we have to use the partial fraction expansion of $\coth\lambda v_0$. We obtain

$$x = \frac{A(\frac{3}{2})^{1/3} r_1^{-1/3}}{v_0} \left\{1 + \frac{2\pi r^{2/3} r_1^{1/3}}{v_0} \sum_{n=0}^{\infty} n I_{-2/3}\left(\frac{n\pi r}{v_0}\right) K_{1/3}\left(\frac{n\pi r_1}{v_0}\right)\right\}, \quad (8)$$

for $0 < r < r_1$.

To summarize, we have the integral representations (3) and (4) of the coordinates together with representations (6), (7) and (8) for x as a function of r on the straight portions of the bounding streamline. In each case the constant A is given by (5).

We are now able to calculate the drag coefficient C_D. This we define in the usual way as

$$C_D = D/\tfrac{1}{2}\rho_1 U_1^2,$$

where D is the drag on the upper surface of the wedge. The local pressure coefficient is defined by

$$C_p = (p - p_1)/\tfrac{1}{2}\rho_1 U_1^2. \quad (9)$$

Then for small δ we have

$$C_D = \delta \int_0^1 C_p \, dx, \quad (10)$$

the integral being evaluated along the side of the wedge. According to the linearized theory it can be shown that

$$C_p = \frac{2}{\gamma+1}(u - u_1) = \frac{2}{\gamma+1}\left(\frac{3}{2}\right)^{2/3}(r^{2/3} - r_1^{2/3}). \quad (11)$$

Hence

$$C_D = -2\left(\frac{3}{2}\right)^{2/3} \frac{\delta}{\gamma+1}\left\{\int_0^{\infty} r^{2/3}\left(\frac{\partial x}{\partial r}\right)_{v=v_0} dr + r_1^{2/3}\right\}.$$

The procedure is now to substitute the series representations (6) and (7) for x. The expression for C_D is obtained as an infinite series. Details of the algebra are suppressed but the final form is

$$\frac{(\gamma+1)^{1/3}C_D}{\delta^{5/3}} = 3^{2/3}\pi^{\frac{1}{2}}\Gamma(\tfrac{1}{3})\left\{\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{2n}n!}\times\right.$$
$$\left.\times\,\Gamma(n+\tfrac{5}{6})(2^{2n+5/3}-1)\zeta(2n+\tfrac{5}{3})(3r_1/2v_0)^{2n}\right\}^{-1}-2(3r_1/2v_0)^{2/3}. \quad (12)$$

This may be written in terms of M_1 through the relation

$$\frac{3r_1}{2v_0}=\frac{(1-M_1^2)^{3/2}}{\delta(\gamma+1)}.$$

For values of the free stream Mach number little different from unity the drag coefficient may be expanded in powers of $1-M_1^2$. The leading terms of such an expansion are

$$C_D=\frac{1\cdot89\delta^{5/3}}{(\gamma+1)^{1/3}}-\frac{2\delta(1-M_1^2)}{\gamma+1}+\frac{0\cdot49(1-M_1^2)^3}{\delta^{1/3}(\gamma+1)^{7/3}}+O\left\{\frac{(1-M_1^2)^6}{\delta^{7/3}(\gamma+1)^{13/3}}\right\}. \quad (13)$$

This expression may be compared with that obtained by Cole. As is to be expected, there is a fixed non-zero limit of C_D as $M_1\to 1$. This value is discussed in more detail in the next section.

In figure 1 the point C is the location of the end of the sonic line and the point where the wake becomes parallel to the free stream. The x-coordinate of C is obtained by letting $r\to 0$ in (8) and has the value

$$x_c=\frac{A(\frac{3}{2})^{1/3}r_1^{-1/3}}{v_0}\left\{1+\frac{2^{5/3}}{\Gamma(\frac{1}{3})}\sum_{n=1}^{\infty}\left(\frac{n\pi r_1}{v_0}\right)^{1/3}K_{1/3}\left(\frac{n\pi r_1}{v_0}\right)\right\}.$$

Finally we can calculate H, the semi-width of the wake. This is given by $H=\delta+h$ where we have to evaluate

$$h=(\gamma+1)^{-1}\int v\,dx,$$

the integral being taken along the sonic line BC. Alternatively we can find H by taking a control surface consisting of the streamline $EOBCD$ and a second streamline at a large distance from the wedge. The momentum principle then gives

$$p_1H=\delta\int_0^1 p\,dx+p_s(H-\delta).$$

The first term on the right-hand side is the force in the x-direction exerted by the wedge OB and the second term that exerted by the streamline BC on the fluid. p_s is the pressure corresponding to sonic velocity. By means of (9), (10) and (11) we obtain

$$H=\delta+\tfrac{1}{2}(\gamma+1)C_D/u_1.$$

This shows that the width of the wake is infinite when the Mach number of the free stream is 1.

3. The limiting case of the sonic free stream

We now consider in more detail the limiting case of the solution discussed in the previous section when the Mach number at infinity tends to 1. The point C of figure 1 now moves to infinity and we have the flow of a gas with $M_1 = 1$ past a wedge where streamlines at constant (sonic) velocity break away from the shoulder and extend to infinity downstream. This is the compressible analogue of the Kirchhoff–Helmholtz flow past a wedge in the theory of incompressible flow. Although we can solve this problem for compressible flow in terms of solutions of the exact hodograph equations the relative simplicity arising from the use of Bessel functions means that information can be obtained with very much less labour although at the cost of whatever loss of accuracy is inherent in the use of the transonic approximation. In this connection it should be noted in particular that the approximation enforced by the stagnation condition at the tip of the wedge is an unsatisfactory feature.

Certain of the results we shall derive may be obtained by letting $r_1 \to 0$ in formulae of the previous section. It is necessary, however, to exercise some caution in performing this operation because of the special nature of the singularity in the hodograph plane corresponding to a free stream Mach number $M_1 = 1$ to which reference has already been made. For example, if we let $r_1 \to 0$ in (3), we obtain

$$y = Br^{1/3} \int_0^\infty \frac{\sinh \lambda(v_0 - v) \,.\, \lambda^{4/3}}{\sinh \lambda v_0} J_{1/3}(\lambda r)\, d\lambda,$$

where B is a constant. It is easy to see that the necessary boundary conditions on $r = 0$ and $v = v_0$ are satisfied by this expression but it is not clear that $y = 0$ when $v = 0$ as the integral does not converge here. It is possible to rewrite this integral as a contour integral and then as a series which gives an analytic continuation of the solution for $v = 0$. However, it is more natural to start from the series form as this may be written down immediately from the analogous incompressible problem, and then to express this as a contour integral. The contour integral thus provides the link between the series and real integral forms of solution. This approach also obviates certain convergence difficulties associated with the derivation of the expression for the drag. In the series form, the solution of this free streamline model has been discussed by Imai (1952). It will be seen that an approximation made in Imai's computation is in fact unnecessary.

If we make use of the standard methods of Kirchhoff and Helmholtz, the usual procedure in the hodograph plane in incompressible flow leads to the result

$$\Psi = k'' \sum_{n=1}^{\infty} n(q/q_1)^{n\pi/\delta} \sin(n\pi\theta/\delta). \tag{14}$$

This solution represents flow which is uniform at infinity with velocity q_1 and which has as a zero streamline a wedge of semi-angle δ whose axis is parallel to the direction of flow. Downstream from the shoulder of the wedge the streamline is one of constant velocity q_1. k'' is a (positive) constant scale factor which is generally chosen to make the length of the wedge unity.

As previously observed, equation (14) can be generalized to include the solutions of the full hodograph equation and the physical problem just described can be solved completely for any subsonic or sonic velocity q_1 (Mackie & Pack 1955). However we shall restrict attention here to performing the same operations with reference to the solutions of Tricomi's equation. This consists of replacing $q^{n\pi/\delta}$ where it occurs by $r^{1/3}K_{1/3}(n\pi r/v_0)$, where $v_0 = (\gamma+1)\delta$. Accordingly we can write

$$y = k'' \sum_{n=1}^{\infty} n \frac{r^{1/3}\,K_{1/3}(n\pi r/v_0)}{r_1^{1/3}\,K_{1/3}(n\pi r_1/v_0)} \sin\frac{n\pi v}{v_0}.$$

Now if $r_1 = 0$ a considerable simplification results, corresponding to the case when $M_1 = 1$. Since $\lim_{z\to 0} z^{1/3}K_{1/3}(z)$ is a non-zero constant the expression becomes

$$y = k' \sum_{n=1}^{\infty} n^{4/3}r^{1/3}K_{1/3}(n\pi r/v_0)\sin(n\pi v/v_0), \tag{15}$$

where again the precise value of k' is as yet unimportant as it will be fixed later.

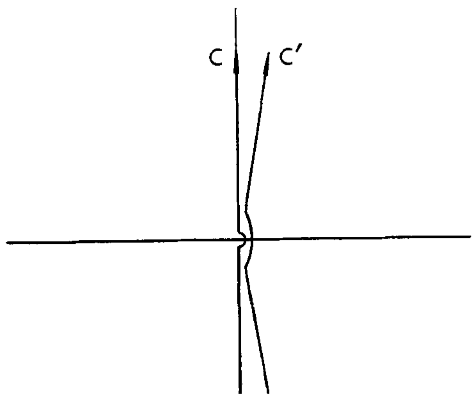

Figure 3. The complex ν-plane.

This series converges for $r > 0$, that is for $M < 1$, but not when $r = 0$. To examine (15) in more detail we rewrite it as

$$y = \mathscr{I}(W) = -\mathscr{I}(k') \sum_{n=1}^{\infty} n^{4/3}r^{1/3}K_{1/3}(n\pi r/v_0)e^{-in\pi v/v_0}. \tag{16}$$

We now express W as a contour integral in the form

$$W = \frac{k'}{2i}\int_C \nu^{4/3}r^{1/3}K_{1/3}(\nu\pi r/v_0)\sin^{-1}\nu\pi e^{i\nu\pi(1-v/v_0)}\,d\nu. \tag{17}$$

C is the contour in the complex ν-plane which goes from $-i\infty$ to $i\infty$, indented at the origin as shown in figure 3. The equivalence of (16) and (17) can be verified by completing the right-hand semi-circle and using the asymptotic properties of $K_{1/3}(z)$ combined with the Cauchy theory of residues. The integrand has a branch point at the origin but is one-valued in the whole plane cut along the negative real axis. We shall also introduce at this stage the contour C' (figure 3) at every point of which $\mathscr{R}(\nu) > 0$, but which is such that it lies to the left of all the singularities of the integrand for which $\mathscr{R}(\nu) > 0$.

If now we write $W = W_1 + W_2$ where W_1 comprises the part of the integral from 0 to $i\infty$, we can rewrite (17) as a real integral. In fact we obtain

$$W = \frac{k'}{2i} \int_0^\infty \{e^{2i\pi/3} t^{4/3} r^{1/3} K_{1/3}(e^{i\pi/2} t\pi r/v_0) e^{-t\pi(1-v/v_0)} - \\ - e^{-2i\pi/3} t^{4/3} r^{1/3} K_{1/3}(e^{-i\pi/2} t\pi r/v_0) e^{t\pi(1-v/v_0)}\} \sinh^{-1} t\pi \, dt.$$

Making use of the formula

$$K_{1/3}(e^{i\pi/2} z) = 3^{-1/2}\pi\{e^{-i\pi/6} J_{-1/3}(z) - e^{i\pi/6} J_{1/3}(z)\},$$

we obtain after some algebra

$$y = \mathscr{I}(W) = \tfrac{1}{2}k'\pi \int_0^\infty r^{1/3} t^{4/3} \sinh t\pi(1 - v/v_0) \sinh^{-1} t\pi J_{1/3}(t\pi r/v_0) \, dt.$$

With the substitution $t = v_0 \lambda/\pi$, this becomes

$$y = \tfrac{1}{2}k'\pi(v_0/\pi)^{7/3} r^{1/3} \int_0^\infty \lambda^{4/3} \sinh \lambda(v_0 - v) \sinh^{-1} \lambda v_0 J_{1/3}(\lambda r) \, d\lambda.$$

Comparison with (3) shows that this is precisely the same expression as would be obtained by letting $r_1 \to 0$ in the work of the previous section. We can now write

$$y = k r^{1/3} \int_0^\infty \lambda^{4/3} \sinh \lambda(v_0 - v) \sinh^{-1} \lambda v_0 J_{1/3}(\lambda r) \, d\lambda, \tag{18}$$

where

$$k = (\tfrac{9}{2})^{1/3} \frac{\pi^{1/2} v_0^{5/3}}{\Gamma(\frac{5}{6})(2^{5/3} - 1)\zeta(\frac{5}{3})}. \tag{19}$$

The value of k is deduced from the limiting cases of (3) and (5) as $r_1 \to 0$ but may also be obtained independently.

Equations (18) and (19) together give the solution past a wedge of unit length with a free streamline breaking away from the shoulder at sonic velocity.

Since this result forms the basis of the work in the following section we shall discuss the solution in more detail. First we note that (15), the imaginary part of (17), and (18), are all representations of the same solution, subject to the suitable correlation of the constants k and k'. However (15) does not converge when $r = 0$ for any v while (18) does not converge when $v = 0$ for any r. By a suitable choice of C', the contour in the complex ν-plane, we can make the integral expression in (17) convergent everywhere except at $r = 0$, $v = 0$ where it must have a singularity. One way to do this is to choose C' to be the contour consisting of the pair of straight lines $\arg(\nu) = \pm(\frac{1}{2}\pi - \epsilon)$ where $0 < \epsilon < \frac{1}{2}\pi$, indented if required at 0. The distortion of the contour from C to C' is easily shown to be valid for some range $r > 0$, $v > 0$ and the extension to other values of r and v follows from an appeal to the principle of analytic continuation.

The value of the x-coordinate in terms of r and v can be obtained by the methods of the previous section in either of the forms

$$x = -(3/2)^{1/3}k'r^{2/3}\sum_{n=1}^{\infty} n^{4/3}K_{2/3}(n\pi r/v_0)\cos(n\pi v/v_0), \tag{20}$$

$$x = (3/2)^{1/3}kr^{2/3}\int_0^{\infty}\lambda^{4/3}\cosh\lambda(v_0-v)\sinh^{-1}\lambda v_0\, J_{-2/3}(\lambda r)\,d\lambda. \tag{21}$$

The pressure can be obtained similarly. However, since these expressions again show non-convergence either when $r = 0$ or $v = 0$, we shall indicate how a rigorous formulation of the drag coefficient can be obtained by means of the contour integral representation.

We have

$$C_D = 2\delta(\gamma+1)^{-1}\int_0^L u\,dx \bigg/ \int_0^L dx,$$

where L is the length of the wedge, and so

$$C_D = 2\delta(\gamma+1)^{-1}\int_0^{\infty} u^2 y_v\,du \bigg/ \int_0^{\infty} u y_v\,du.$$

If now we replace y_v by the appropriate contour integral derived from (17), we can invert the order of integration and obtain expressions such as $[z^{4/3}K_{4/3}(z)]_0^{\infty}$, $[z^{2/3}K_{2/3}(z)]_0^{\infty}$. The reason for replacing C by C' now becomes clear. For since we have performed the integrations with respect to u (or r) first, we must have z in the above expressions with a positive real part and this can only be done if $\mathscr{R}(\nu) > 0$ on C'. After some algebra we obtain

$$C_D = 2\,.\,3^{2/3}\Gamma(\tfrac{4}{3})\delta(\gamma+1)^{-1}\left(\frac{v_0}{\pi}\right)^{2/3}\int_{C'}\frac{d\nu}{\sin\nu\pi}\left\{\int_{C'}\frac{\nu^{2/3}\,d\nu}{\sin\nu\pi}\right\}^{-1}.$$

C' can now be replaced by C and the resulting contour integrals are standard forms for functions associated with Riemann zeta functions. Further algebra now reduces the expression for C_D to exactly that obtained when r_1 is put equal to zero in (12). Numerically we have

$$C_D = 1{\cdot}89\delta^{5/3}(\gamma+1)^{-1/3}. \tag{22}$$

Some explanation should be given regarding the discrepancy between the factor 1·89 appearing in (22) and the numerical results obtained by Imai (1952) and by Mackie & Pack (1955). There is an error in computation in Imai's work, and in addition he replaces $K_{1/3}(z)$ by its asymptotic form in assessing the drag whereas in fact this is not necessary as the expressions can be integrated exactly. Mackie & Pack obtain the drag coefficient for small δ as the leading term of the expression for general δ when the velocity is sonic. Although the details are not given, the analytic expression is the same as that given by the leading term in (12) as $r_1 \to 0$ but the numerical factor is subsequently computed incorrectly and this accounts for the discrepancy.

Comparison with the more elaborate computations of Guderley & Yoshihara (1950) based on a solution valid up to the limiting Mach wave shows the pressure obtained in the present solution to be slightly higher. Figure 4 shows the variation of pressure along the wedge face compared with that calculated by Guderley & Yoshihara and the variation is seen to be slight. For application in the following section it is desirable to have some estimate of the value of r along the wedge face and this is shown by plotting r against x in figure 5.

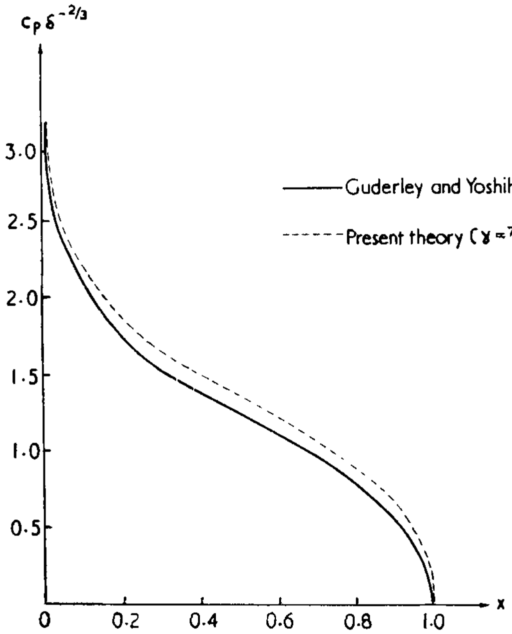

Figure 4. Chordwise pressure distribution.

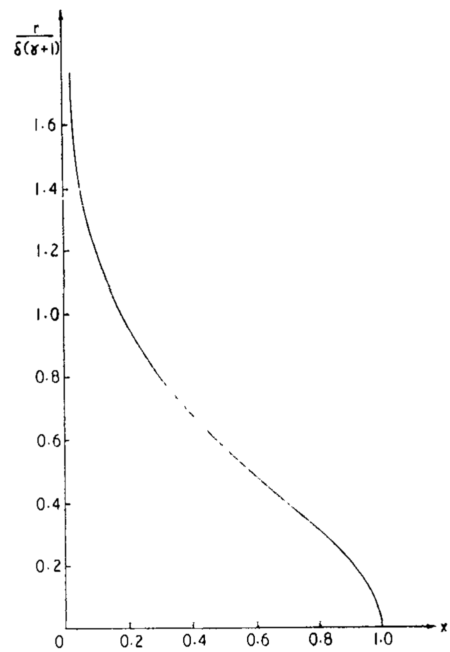

Figure 5. Chordwise variation of r.

4. The perturbation problem and its solution

We now consider the flow past a symmetric wedge-type profile of arbitrary shape provided this is such as may be represented in the physical plane by a small perturbation upon the wedge with straight sides. As in the previous section, the Mach number of the flow at infinity upstream is 1.

The pattern in the physical plane is much as in figure 1 except that the line OB is replaced by a curved line lying close to it. We shall take the semi-angle at the tip of the new body to be δ as before but the slope of the profile at the shoulder will now be δ'. The equation of the boundary is taken to be $y = \delta x + \epsilon F(x)$. In the subsequent work we shall neglect terms which are $O(\epsilon^2)$. The streamline $\Psi' = 0$ downstream from the body will again be presumed to be one of constant velocity which is the velocity of sound.

We now set up the boundary value problem in the hodograph plane as shown in figure 6. $y(u, v)$ will have a given singularity at the origin DE and vanishes on EO and on BD. B is the point $(0, v_0')$ where $v_0' = (\gamma+1)\delta'$. Further, $y = 0$ on OB where OB is some unspecified curve into which the body profile is mapped. Its equation is of the form $v = v_0 + \epsilon f(r)$, where we recall that r is related to u by (2). Finally, there is the stagnation condition $y = 0$, $x = 0$ as $r \to \infty$.

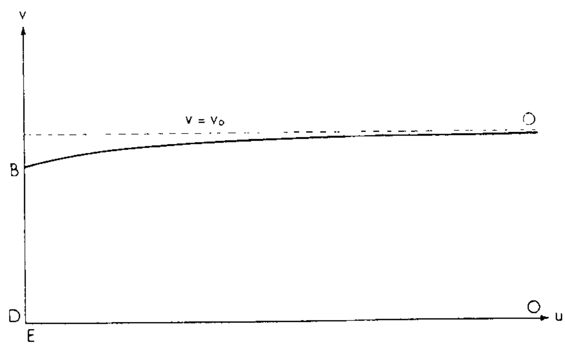

Figure 6. The hodograph plane for the perturbation problem.

$f(r)$ can now be found. Since OB is a streamline it follows that along it

$$\frac{dy}{dx} = \delta + \epsilon F'(x) = \theta = \frac{v}{\gamma+1} = \delta + \frac{\epsilon f(r)}{\gamma+1}.$$

Thus

$$[F'(x)]_{x = x(r,v)} = (\gamma+1)^{-1} f(r) \quad \text{on } v = v_0 + \epsilon f(r). \tag{23}$$

We now write

$$x = x^* + \epsilon x_p, \qquad y = y^* + \epsilon y_p,$$

where $x^*(r, v)$, $y^*(r, v)$ are the values of x and y in the unperturbed flow obtained in the previous section for given values of r and v. If this is substituted in (23) we obtain

$$f(r) = (\gamma+1) F'\{x^*(r, v_0)\} + O(\epsilon). \tag{24}$$

The function $x^*(r, v_0)$ is given by (20) or (21) and is shown graphically in figure 5. The shape of the streamline in the hodograph plane is now given to the first order in ϵ. This result is important for it enables the problem to be set up in the hodograph plane directly from a given configuration in the physical plane. Thus it avoids the disadvantage of the *a posteriori* determination of the boundary streamline to which reference was made in § 1.

The boundary condition on OB requires that $y(r, v) = 0$ on $v = v_0 + \epsilon f(r)$ That is

$$y^*\{r, v_0 + \epsilon f(r)\} + \epsilon y_p\{r, v_0 + \epsilon f(r)\} = 0.$$

Since $y^*(r, v_0) = 0$ we must have

$$y_p(r, v_0) = -f(r)[\partial y^*/\partial v]_{v=v_0}. \tag{25}$$

This function on the right-hand side, which is known when the shape of the profile is known, we denote by $H(r)$.

In seeking a solution of the perturbation problem we must therefore find a function y_p which vanishes on $v = 0$ for all u, on $u = 0$ for $0 < v < v_0$, and which takes the value $H(r)$ on $v = v_0$. y_p is non-singular at the origin as the doublet-type singularity necessary for the flow is already contained in y^*. Finally y_p must not upset the stagnation condition at $x = 0$, $y = 0$.

All the conditions except (25) are immediately satisfied by a solution of the form

$$y_p = \int_0^\infty G(\lambda) r^{1/3} J_{1/3}(\lambda r) \sinh \lambda v \, d\lambda. \tag{26}$$

The condition (25) is satisfied if

$$\int_0^\infty G(\lambda) \sinh \lambda v_0 \, r^{1/3} J_{1/3}(\lambda r) \, d\lambda = H(r),$$

and it follows from the Hankel inversion theorem that

$$G(\lambda) = \frac{\lambda}{\sinh \lambda v_0} \int_0^\infty H(t) t^{2/3} J_{1/3}(\lambda t) \, dt. \tag{27}$$

The complete solution of the perturbation problem is now $y = y^* + \epsilon y_p$, where y^* is given by (18) and y_p by (26) and (27). The corresponding value of x follows from equations (1). It is $x = x^* + \epsilon x_p$, where x^* is given by (21) and

$$x_p = -(\tfrac{3}{2})^{1/3} \int_0^\infty G(\lambda) r^{2/3} J_{-2/3}(\lambda r) \cosh \lambda v \, d\lambda.$$

An expression is now found for the drag coefficient on the upper half of the profile. In estimating this, allowance must be made for the variation in slope of the profile and for the fact that the total length is no longer necessarily unity. The natural extension of the definition is

$$C_D = \int_{r=\infty}^{r=0} C_p[\delta + (\gamma+1)^{-1} \epsilon f(r)] \, ds \Big/ \int_{r=\infty}^{r=0} ds,$$

where the integration is taken along the side of the profile and the limits indicate that it is carried out from the tip to the shoulder. Now on the profile

$$x = x^* + \epsilon x_p, \qquad y = \epsilon y_p.$$

Hence $ds = dx\{1 + O(\epsilon^2)\}$ and consequently we can replace ds in the integrals by dx and write

$$C_D = \frac{\displaystyle\int_\infty^0 C_p\{\delta + \epsilon(\gamma+1)^{-1} f(r)\} \left\{\frac{\partial x^*}{\partial r} + \epsilon \frac{\partial x_p}{\partial r}\right\}_{v=v_0} dr}{\displaystyle\int_\infty^0 \left\{\frac{\partial x^*}{\partial r} + \epsilon \frac{\partial x_p}{\partial r}\right\}_{v=v_0} dr}.$$

The terms independent of ϵ in the numerator and denominator are respectively the drag coefficient of the original straight wedge given by (22) which we shall here call C_D^*, and its length which is unity. Then if $C_D = C_D^* + \epsilon C_{Dp}$ we have, after some simplification,

$$C_{Dp} = \frac{3}{(\gamma+1)^2}\int_0^\infty rH(r)\,dr - \frac{3\delta}{\gamma+1}\int_0^\infty r\left(\frac{\partial y_p}{\partial v}\right)_{v=v_0} dr + \\ + (\tfrac{3}{2})^{1/3}\, C_D^* \int_0^\infty r^{1/3}\left(\frac{\partial y_p}{\partial v}\right)_{v=v_0} dr. \quad (28)$$

Inspection of (28) shows that three effects contribute to the change in the drag coefficient. The first term represents the effect due to the variation in slope of the profile in that the angle between the direction of the free stream and the direction in which the pressure acts (namely, normal to the surface) varies along the body. The second term is an estimate of the effect of the change in the magnitude of the pressure along the body due to the perturbation of the wedge profile. The third term is a correction due to the change in length of the body and is necessary since C_{Dp} is defined as a dimensionless quantity. We now recall that $H(r)$ is by definition $y_p(r, v_0)$. It follows that the integrands of the three terms in (28) are all of the same order in δ and hence that the terms themselves are in descending order of magnitude for small δ since C_D^* is proportional to $\delta^{5/3}$. It is of interest to note that an assessment of the most important term can be made without obtaining the explicit solution. For from (24) and (25), $H(r)$ depends only on the given perturbation in the physical plane and on the solution of the unperturbed problem. Thus for a given profile, the leading term in the drag coefficient, C_{Dp}, can be quickly evaluated by a numerical integration, that is, by evaluating the integral $\int C_p^*(dy/dx)\,dx^*$ along the wedge, where C_p^* is the pressure coefficient for the unperturbed wedge profile and dy/dx is the true slope of the new profile.

This research has been sponsored in part by the Air Research and Development Command, U.S. Air Force, under contract AF 61(514)–1170, through the European Office A.R.D.C.

References

Cole, J. D. 1951 *J. Math. Phys.* **30**, 79.
Guderley, G. & Yoshihara, H. 1950 *J. Aero. Sci.* **17**, 723.
Imai, I. 1952 *J. Aero. Sci.* **19**, 496.
Lighthill, M. J. 1947 *Proc. Roy. Soc.* A, **191**, 352.
Mackie, A. G. & Pack, D. C. 1955 *J. Rat. Mech. Anal.* **4**, 177.
Roshko, A. 1954 *Nat. Adv. Comm. Aero., Wash., Tech. Note* no. 3168.